Mathematical Challenges in Feynman Path Integrals

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Mathematical Challenges in Quantum Mechanics
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Outline

Introduction: the problem of Feynman integration

Infinite dimensional oscillatory integrals

Recent results and open problems

- Construction techniques and the need of counterterms
- The interpretation of Feynman's dynamics

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Path integrals in theoretical physics

- R. Feynman and A. Hibbs, *Quantum mechanics and path integrals*. Dover (2010)
- B. Simon, *Functional integration and quantum physics.* Academic Press, (1979).
- J. Glimm and A. Jaffe, *Quantum physics. A functional integral point of view*. Springer-Verlag, (1987).
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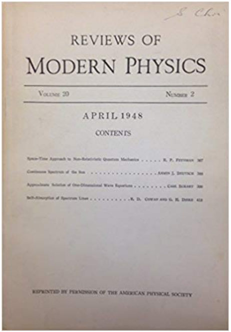
- H. Kleinert, *Path integrals in quantum mechanics, statistics, polymer physics, and financial markets.* World Scientific (2009)
- C. Grosche, *Path integrals, hyperbolic spaces and Selberg trace formulae*. World Scientific (2013).
- ...

Origins: Feynman's formulation of quantum dynamics

$$\begin{cases} i\hbar\frac{\partial}{\partial t}\psi(t,x) = -\frac{\hbar^2}{2m}\Delta\psi(t,x) + V(x)\psi(t,x) \\ \psi(0,x) = \psi_0(x) \end{cases}$$

$$\psi(t,x) = \quad \text{“}\frac{1}{C} \quad \int_{\{\gamma|\gamma(t)=x\}} e^{\frac{i}{\hbar}S(\gamma)}\psi_0(\gamma(0))D\gamma \quad \text{”}$$


REVIEWS OF

MODERN PHYSICS

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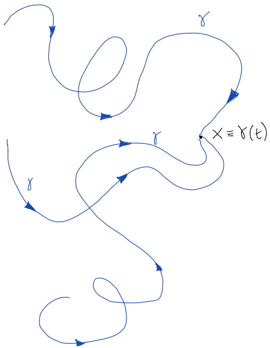
Feynman path integrals

$$\psi(t,x) = \text{“}\frac{1}{C}\int_\Gamma e^{\frac{i}{\hbar}S(\gamma)}\psi_0(\gamma(0))D\gamma\text{”}$$

$$\Gamma := \{\gamma : [0,t] \to \mathbb{R}^d : \gamma(t) = x\}$$

$$S(\gamma) = \int_0^t \mathcal{L}(\gamma(s), \dot{\gamma}(s))ds = \int_0^t \left(\frac{m}{2}\dot{\gamma}^2(s) - V(\gamma(s))\right) ds$$

$$C = \int_\Gamma e^{\frac{i}{\hbar}S(\gamma)}D\gamma$$



Feynman path integrals

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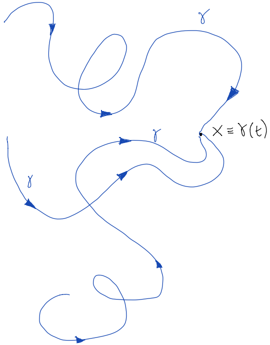
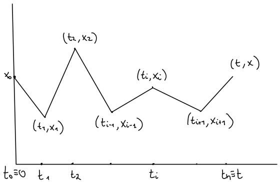


Quantization technique

$$\mathcal{L}(q,\dot q) \quad \mapsto \quad \psi(t) = U(t)\psi_0$$

How to compute Feynman integrals: piecewise linear approximations



$$
\begin{aligned}
S(\gamma) &= \int_0^t \frac{1}{2}\dot{\gamma}(s)ds - \int_0^t V(\gamma(s)ds \\
&\sim \sum_{j=1}^{n} \frac{1}{2}\frac{(x_j - x_{j-1})^2}{(t_j - t_{j-1})} \\
&\quad - \sum_{j=1}^{n} V(x_j)(t_j - t_{j-1})
\end{aligned}
$$

$$
\frac{1}{C}\int_{\gamma(t)=x} e^{\frac{i}{\hbar}S(\gamma)}\psi_0(\gamma(0))d\gamma
$$

$$
:= \lim_{n\to\infty}\int_{\mathbb{R}^n} \frac{e^{\frac{i}{\hbar}\sum_{j=1}^{n}\left(\frac{(x_j-x_{j-1})^2}{(t/n)^2} - V(x_j)\right)\frac{t}{n}}}{(2\pi i\hbar t/n)^{n/2}}\psi_0(x_0)dx_0\dots dx_{n-1}
$$

Formal derivation of Feynman's formula: Trotter product formula for $U(t) = e^{-\frac{i}{\hbar}Ht}$, $H = \frac{-\hbar^2\Delta}{2} + V$

$$\psi(t,x) = e^{-\frac{it}{\hbar}(\frac{-\Delta}{2}+V)}\psi_0(x) = \lim_{n\to\infty}\left(e^{\frac{it\hbar\Delta}{2n}}e^{-\frac{it}{n\hbar}V}\right)^n\psi_0(x), \text{ a.e.} x \in \mathbb{R}$$

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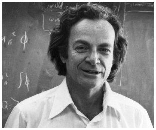
&= {}''\frac{1}{C}\int_{\gamma(t)=x} e^{\frac{i}{\hbar}S(\gamma)}\psi_0(\gamma(0))d\gamma''
\end{aligned}$$

What is a Feynman path integral?

$$\frac{1}{C} \quad \int_\Gamma e^{\frac{i}{\hbar}S(\gamma)}\psi_0(\gamma(0))D\gamma \stackrel{?}{=} \int_\Gamma f(\gamma)d\mu_F(\gamma)$$



. . . one feels as Cavalieri must have felt calculating the volume of a pyramid before the invention of calculus. R. P. FEYNMAN

Mathematical definition of Feynman integrals

Can Feynman integrals be realized as Lebesgue integrals on path space Γ with respect to a Feynman measure μ_F ?

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Why?

- Continuity properties of Lebesgue integrals:
  $\lim_{n\to\infty} f_n(\gamma) = f(\gamma), \quad |f_n| \leq g \in L^1(d\mu) \quad \Rightarrow$
  $\lim_{n\to\infty} \int_\Gamma f_n \, d\mu_F = \int_\Gamma f \, d\mu_F$

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  $\lim_{n\to\infty} \int_\Gamma f_n\, d\mu_F = \int_\Gamma f\, d\mu_F$
- The piecewise linear approximation presents ambiguities even in simple cases!!! Example: charge particle in an external magnetic field

The Schrödinger equation with magnetic field $B = \nabla A$

$$i\hbar \frac{\partial}{\partial t}\psi(t,x) = \frac{1}{2}\left(-i\hbar\nabla - A(x)\right)^2 \psi(t,x),$$

$$\psi(t,x) = \int_{\gamma(t)=x} e^{\frac{i}{2\hbar}\int_0^t |\dot\gamma(s)|^2 ds + \frac{i}{\hbar}\int_0^t A(\gamma(s))\cdot\dot\gamma(s) ds}\psi_0(\gamma(0))d\gamma,$$

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Problem with the piecewise linear approximations

$$\int e^{\frac{i}{2\hbar}\sum_i \frac{|\gamma(t_{i+1})-\gamma(t_i)|^2}{t_{i+1}-t_i} + \frac{i}{\hbar}\sum_i A(\gamma(\tilde t_i))\cdot(\gamma(t_{i+1})-\gamma(t_i))}$$

$$\psi_0(\gamma(0))\prod_i \frac{d\gamma(t_i)}{(2\pi i\hbar(t_{i+1}-t_i))^{1/2}},$$

$0 = t_0 < t_1 < t_2 < \cdots < t_n = t$, $\tilde t_i \in [t_i, t_{i+1}]$, $i = 0, \ldots, n-1$.
different choices of the point $\tilde t_i \in [t_i, t_{i+1}]$ lead to different results!!!

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correct choice: mid-point rule $\tilde t_i \equiv \frac{t_{i+1}+t_i}{2}$

.

Construction of Feynman measure

$$\mu_F(d\gamma) = " \frac{e^{\frac{i}{\hbar}S(\gamma)}D\gamma}{\int_\Gamma e^{\frac{i}{\hbar}S(\gamma)}D\gamma}$$

Problems

1. Integration theory on an ∞-dim space

$$\Gamma = \{\gamma : [0,t] \to \mathbb{R} : \gamma(t) = x\}$$

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Problems

1. Integration theory on an ∞-dim space

$$\Gamma = \{\gamma : [0,t] \to \mathbb{R} : \gamma(t) = x\}$$

2. Presence of the imaginary unity i in the formal integrand $e^{\frac{i}{\hbar}S_t(\gamma)}$ and in its "finite dimensional approximations"

$$\frac{e^{\frac{i}{\hbar}\sum_{j=1}^{n}\frac{1}{2}\frac{(x_j-x_{j-1})^2}{(t/n)}}}{(2\pi i\hbar t/n)^{n/2}}$$

The Feynman-Kac formula

M. Kac (1949): the solution of the heat equation

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) = \frac{1}{2}\Delta u(t,x) - V(x)u(t,x), \\ u(0,x) = u_0(x) \end{cases}$$

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$$u(t,x) \quad = \quad e^{(\frac{\Delta}{2} - V)t} u_0(x) = \lim_{n\to\infty} \left(e^{\frac{t}{2n}\Delta} e^{-\frac{t}{n}V}\right)^n u_0(x) \quad \text{a.e. } x \in \mathbb{R}$$

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Signed and complex measures on $\mathbb{R}^{[0,t]}$

Let's μ be the finite additive measure (on $\Omega = \mathbb{R}^{[0,t]}$) defined on the algebra of "cylinder sets" $I_k \subset \Omega = \{x : [0,\infty) \to \mathbb{R}\}$, $I_k := \{\omega \in \Omega : \omega(t_j) \in [a_j, b_j], j = 1, \dots k\}$, $0 < t_1 < t_2 < \dots t_k$,

$$\mu(I_k) = \int_{a_1}^{b_1} \dots \int_{a_k}^{b_k} \prod_{j=0}^{k-1} G_{t_{j+1}-t_j}(x_j, x_{j+1}) dx_1 \dots dx_k,$$

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Heat equation $\partial_t u(t) = \frac{\Delta}{2} u(t)$, where $G_t(x,y) = \frac{e^{-\frac{(x-y)^2}{2t}}}{\sqrt{2\pi t}}$
Kolmogorov theorem provides the existence of σ-additive extension of μ

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Schrödinger eq. $i\hbar\partial_t u(t) = -\frac{\hbar^2}{2}\Delta u(t)$: $G_t(x,y) = \dfrac{e^{i\frac{(x-y)^2}{2\hbar t}}}{\sqrt{2\pi i \hbar t}}$

Generalization of Kolmogorov's theorem to signed or complex measures

Theorem (E. Thomas (2001))

Let $(E_j, \Sigma_J, \pi_J^K)_{J \in A}$ *be a projective family of measurable spaces, where* $\pi_J^K : E_K \to E_J$, $J \leq K$. *Let* $\{\mu_J\}_{J \in A}$ *(complex) measures on* (E_J, Σ_J) *satisfying the* compatibility condition

$$\mu_J = \pi_J^K \mu_K, \qquad J \leq K, \quad J, K \in A.$$

A necessary condition for the existence of a (complex) measure μ *on* $\varprojlim_J E_J$ *such that* $\mu_J = \pi_J \circ \mu$ *is*

$$\sup_J |\mu_J| < \infty$$

Non existence of the Feynman measure

$J = \{t_1, t_2, \dots, t_k\} \subset [0, T]$, $\mu_J \equiv \mu_{t_1,t_2,\dots,t_k}$ Borel measure on $\mathbb{R}^J$

$$\mu_{t_1,t_2,\dots,t_k}([a_1, b_1] \times \dots \times [a_k, b_k]) =$$
$$= \int_{a_1}^{b_1} \dots \int_{a_k}^{b_k} \prod_{j=0}^{k-1} G^{z}{}_{t_{j+1}-t_j}(x_{j+1}, x_j) dx_{j+1}, \quad (1)$$

$$G^{z}{}_{t}(x, y) = \frac{e^{-\frac{1}{2tz}|x-y|^2}}{\sqrt{2\pi t z}}, \qquad \sigma \in \mathbb{C}, Re(z) \geq 0$$

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$$|\mu_{t_1,t_2,\ldots,t_k}| = \left(\int_{\mathbb{R}} |G^{z}{}_1(x, 0)| dx\right)^k = \left(\frac{|z|}{Re(z)}\right)^{k/2} \to \infty \text{ if } Im(z) \neq 0$$

there cannot exist a σ-additive complex (resp. signed) **bounded variation** measure μ on $\mathbb{R}^{[0,T]}$ whose cylindrical approximations are given by (1).[R.H. Cameron (1960)]

An alternative integration theory

Realization of the "integral"

$$f \in C_0(X) \;\mapsto\; \int_X f(x)d\mu(x) := I_\mu(f),$$

$$\downarrow$$

$$f \in \mathcal{D} \;\mapsto\; I(f)$$

as a linear (continuous) functional on a "suitable" (topological) vector space $\mathcal{D}$ of "integrable functions".

$$\downarrow$$

Development of a theory of *Projective systems of functionals* [1] [2]

[1] S. Albeverio and S.M., *Infinite dimensional oscillatory integrals as projective systems of functionals.* J. Math. Soc. Japan 67 (2015), no. 4.

[2] S. Albeverio and S.M., *A unified view to infinite-dimensional integration*. Rev. Math phys. 28 (2016), no. 2.

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Aim: construction of a linear functional $L : \mathcal{D}_L \to \mathbb{C}$ such that

1. $\mathcal{D}_L$ contains the cylinder functions:

$$f(\gamma) = F(\gamma(t_1), \ldots, \gamma(t_n)), \quad F \in C_b(\mathbb{R}^n), \gamma \in \mathbb{R}^{[0,T]}$$

2.

$$L(f) = \int F(x_1, \ldots, x_n) d\mu_{t_1,t_2,\ldots,t_n}(x_1, \ldots, x_n)$$

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$$f(\gamma) = F(\gamma(t_1), \dots, \gamma(t_n)), \quad F \in C_b(\mathbb{R}^n), \gamma \in \mathbb{R}^{[0,T]}$$

2. $$L(f) = \int F(x_1, \dots, x_n) d\mu_{t_1,t_2,\dots,t_n}(x_1, \dots, x_n)$$

If $\sup_{t_1,\dots,t_n} |\mu_{t_1,t_2,\dots,t_k}| = +\infty$, then L cannot be of the form $L(f) = \int_{\mathbb{R}^{[0,T]}} f d\mu$, with $\mu = \varprojlim \mu_{t_1,\dots,t_n}$, $f \in L^1(\mathbb{R}^{[0,T]}, d\mu)$

Possible definitions of Feynman path integrals

- **Infinite dimensional distribution, White noise calculus** (C. DeWitt-Morette (1972), E. Thomas (1999), T. Hida, L. Streit (1982),...)
- **Sequential approach** (E. Nelson (1964), ..., D. Fujiwara, N. Kumano-Go (2005), F. Nicola and I. Trapasso (2020)...)
- **Analytic continuation of Wiener integrals** (R. Cameron (1960), H. Doss (1980), Kallianpur, Kannan, Karandikar (1985)....)
- **Complex poisson measures** (A.M. Chebotarev, V.P. Maslov (1979)...)
- **Non standard analysis** (S. Albeverio, J.Fenstat, R. Høegh-Krohn, T Lindstrøm (1986), Loo (1999))
- **Infinite dimensional oscillatory integrals** (S. Albeverio, R. Høegh-Krohn (1976), D. Elworthy, A. Truman (1984), S.Albeverio, Z. Brzeźniak (1993),...)

Outline

Oscillatory integrals on $\mathbb{R}^n$

$$\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}\Phi(x)} f(x)dx \tag{2}$$

$$\Phi : \mathbb{R}^n \to \mathbb{R}, \qquad f : \mathbb{R}^n \to \mathbb{C}, \qquad \hbar \in \mathbb{R}^+$$

Examples: Fresnel $\int_{\mathbb{R}} e^{\frac{i}{\hbar}x^2} f(x)dx$, Airy $\int_{\mathbb{R}} e^{\frac{i}{\hbar}x^3} f(x)dx$

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Definition (Hörmander)

if for any $g \in S(\mathbb{R}^n)$, such that $g(0) = 1$ the limit

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}\Phi(x)} g(\epsilon x) f(x)dx$$

exists and is independent of g then it is called **oscillatory integral** of f w.r.t. the phase function Φ and denoted

$$\int_{\mathbb{R}^n}^{o} e^{\frac{i}{\hbar}\Phi(x)} f(x)dx$$

Example : $\int_{\mathbb{R}}^{o} e^{\frac{i}{2}|x|^2} dx = \sqrt{2\pi i}$

Infinite dimensional oscillatory integrals on $(\mathcal{H}, \langle \, , \, \rangle)$

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x,x\rangle} f(x) dx$$

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Definition (D.Elworthy and A.Truman. (1984))

$(\mathcal{H}, \langle \, , \, \rangle)$ real separable Hilbert space,
$f : \mathcal{H} \to \mathbb{C}$ is "Fresnel integrable" if for any $\{P_n\}_{n\in\mathbb{N}}$, with $P_n \leq P_{n+1}$, $P_n \uparrow I_{\mathcal{H}}$ the limit

$$\lim_{n\to\infty} (2\pi i\hbar)^{-n/2} \int_{P_n\mathcal{H}}^{o} e^{\frac{i}{2\hbar}\langle P_n x, P_n x\rangle} f(P_n x) dP_n x$$

exists and is independent of $\{P_n\}$. In this case it is denoted

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x,x\rangle} f(x)dx$$

"Fresnel integrable functions": the Banach algebra $\mathcal{F}(\mathcal{H})$

$$\mathcal{M}(\mathcal{H}) = \{\mu \text{ complex Borel measure on } \mathcal{H} \text{ with finite total variation} |\mu|\}$$

$$\mathcal{F}(\mathcal{H}) = \{f : \mathcal{H} \to \mathbb{C} \colon f(x) = \int_{\mathcal{H}} e^{i\langle x,y\rangle} d\mu_f(y), \text{ for some } \mu_f \in \mathcal{M}(\mathcal{H})\}$$

$(\mathcal{F}(\mathcal{H}), \|;\|)$ Banach algebra, $\|f\| := |\mu_f|$ $f \cdot g(x) := f(x)g(x)$.

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Theorem (Parseval-type equality)

[*Albeverio and Høegh-Krohn (1976), Elworthy and Truman (1984)*]

$$f \in \mathcal{F}(\mathcal{H}) \qquad \widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x,x\rangle} f(x)dx = \int_{\mathcal{H}} e^{-\frac{i\hbar}{2}\langle x,x\rangle} d\mu_f(x)$$

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Remark: Since $|\int_{\mathcal{H}} e^{-\frac{i\hbar}{2}\langle x,x\rangle} d\mu_f(x)| \leq |\mu_f| = \|f\|$ the

$$f \in \mathcal{F}(\mathcal{H}) \mapsto \widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x,x\rangle} f(x)dx$$

is a linear functional continuous in the $\mathcal{F}(\mathcal{H})$-norm!

Solution of Schrödinger equation

Cameron-Martin space $(\mathcal{H}_t, \langle\ ,\ \rangle)$

$\mathcal{H}_t = \{\gamma : [0,t] \to \mathbb{R}^d, |\, \gamma(t) = 0, \int_0^t |\dot{\gamma}(s)|^2 ds < \infty\}$,

$\langle \gamma_1, \gamma_2 \rangle = \int_0^t \dot{\gamma}_1(s) \cdot \dot{\gamma}_2(s) ds$

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$$\begin{cases} i\hbar \frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\Delta\psi + V\psi \\ \psi(0) = \psi_0, \end{cases} \tag{3}$$

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$$\begin{cases} i\hbar \frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\Delta\psi + V\psi \\ \psi(0) = \psi_0, \end{cases} \tag{3}$$

Theorem (Albeverio and Høegh-Krohn (1976))

Let $V \in \mathcal{F}(\mathbb{R}^d)$ and $\psi_0 \in L^2(\mathbb{R}^d) \cap \mathcal{F}(\mathbb{R}^d)$. Then the map

$$f(\gamma) = e^{-\frac{i}{\hbar} V(\gamma(s)+x)ds)} \psi_0(\gamma(0)+x), \qquad \gamma \in \mathcal{H}_t$$

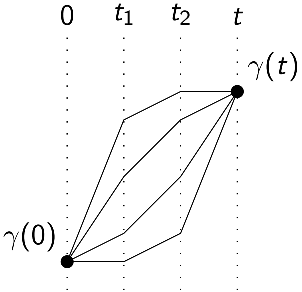
belongs to $\mathcal{F}(\mathcal{H}_t)$ and its infinite dimensional Fresnel integral

$$\psi(t,x) = \widetilde{\int_{\mathcal{H}_t}} e^{\frac{i}{2\hbar}\langle\gamma,\gamma\rangle} e^{-\frac{i}{\hbar} V(\gamma(s)+x)ds)} \psi_0(\gamma(0)+x) d\gamma$$

provides a representation for the solution of (3).

The piecewise linear approximations

For $n \geq 1$, consider the partition $0 < t_1 < \dots < t_k < \dots < t$, $t_k = k\frac{t}{n}$, $k = 0, \dots, n$.
Let $H_n \subset \mathcal{H}_t$ be the finite dimensional subspace of *piecewise linear paths*



$$\gamma(s) = \sum_{k=1}^{n} \chi_{[t_{k-1},t_k]}(s) \left(\gamma(t_{k-1}) + \frac{\gamma(t_k) - \gamma(t_{k-1})}{t_k - t_{k-1}} (s - t_{k-1}) \right)$$

Let $P_n : \mathcal{H}_t \to \mathcal{H}_t$ the projector operator onto H_n.
By setting:

$$\gamma_{x_0,\dots,x_{n-1}}(s) = \sum_{k=1}^{n} \chi_{[t_{k-1},t_k]}(s) \left(x_{k-1} + \frac{x_k - x_{k-1}}{t_k - t_{k-1}}(s - t_{k-1}) \right)$$

we have

$$\lim_{n\to\infty} \frac{\int^{o}_{P_n\mathcal{H}} e^{i\frac{\|P_n\gamma\|^2}{2\hbar}} f(P_n\gamma) d(P_n\gamma)}{\int^{o}_{P_n\mathcal{H}} e^{i\frac{\|P_n\gamma\|^2}{2\hbar}} d(P_n\gamma)}$$

$$= \lim_{n\to\infty} \frac{\int^{o}_{\mathbb{R}^{nd}} e^{\frac{i}{\hbar}\sum_{j=1}^{n} \frac{(x_j - x_{j-1})^2}{2(t/n)^2}} f(\gamma_{x_0,\dots,x_{n-1}}) dx_0 \cdots dx_{n-1}}{(2\pi i\hbar t/n)^{nd/2}}$$

Applications of Infinite dimensional Fresnel integrals

- Schrödinger equation with polynomially growing potentials [Albeverio and Mazzucchi 2005]
- Quantum theory of open systems and stochastic Schrödinger equation. [Albeverio, Guatteri and Mazzucchi 2002]
- Relativistic quantum boson field (free or with regularized bounded interaction) [Albeverio and Høegh-Krohn 1976]
- Chern-Simons functional integral [Albeverio and Schäfer 1995, Albeverio and Sengupta 1997]

Outline

Introduction: the problem of Feynman integration

Infinite dimensional oscillatory integrals

Recent results and open problems

- Construction techniques and the need of counterterms
- The interpretation of Feynman's dynamics

Feynman path integrals for the magnetic field. A perturbative approach

S. Albeverio, N. Cangiotti, S.M. *Rigorous mathematical construction of Feynman path integrals for the Schrödinger equation with magnetic field*. Comm. Math. Phys. (2020).

$$\psi(t,x) = \int e^{\frac{i}{2\hbar}\int_0^t \|\dot\gamma(s)\|^2 ds - \frac{i}{\hbar}\int_0^t \lambda \mathbf{a}(\gamma(s)+x)\cdot\dot\gamma(s)ds}\psi_0(\gamma(t)+x)d\gamma$$

$$:= \sum_n \frac{-i\lambda^n}{n!}\int_{\mathcal{H}_t} e^{\frac{i}{2\hbar}\int_0^t \|\dot\gamma(s)\|^2 ds}\left(\int_0^t \mathbf{a}(\gamma(s)+x)\cdot\dot\gamma(s)ds\right)^n \psi_0(\gamma(t)+x)d\gamma$$

Assumptions:

$a_i(x) = \int_{\mathbb{R}^3} e^{ix\cdot k} d\mu_i(k),\ i = 1,2,3,$
$\psi_0(x) = \int_{\mathbb{R}^3} e^{ix\cdot k}\hat\psi_0(k)dk,$
with μ_i, $i = 1,2,3$, and $\hat\psi_0$ compactly supported

An extension of the class of integrable functions

Problem: even if we assume that $a_i \in \mathcal{F}(\mathbb{R}^d)$ for all $i = 1, 2, 3$, the map

$$\gamma \in \mathcal{H}_t \mapsto \int_0^t \mathbf{a}(\gamma(s))\dot{\gamma}(s)ds$$

does not belong to $\mathcal{F}(\mathcal{H}_t)$ (apart from trivial cases)

$$\Downarrow$$

Parseval type equality cannot be used!

$$\Downarrow$$

A different computation technique has to be developed

Oscillatory vs Gaussian integrals

Main idea: a deformation of the integration contour in the complex plane

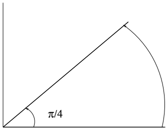


Figure 1

$$\int_{\mathbb{R}} f(x) \frac{e^{\frac{i}{\hbar}\frac{x^2}{2}}}{\sqrt{2\pi i \hbar}} dx \quad = \quad \int_{\mathbb{R}} f(e^{i\pi/4}\sqrt{\hbar}x) \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx$$

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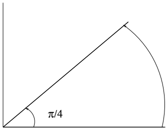

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$$\int_{\mathbb{R}^n} f(x) \frac{e^{\frac{i}{\hbar}\frac{\|x\|^2}{2}}}{(2\pi i\hbar)^{n/2}} dx = \int_{\mathbb{R}^n} f(e^{i\pi/4}\sqrt{\hbar}x) d\mu_G(x)$$

Infinite dimensional case

- $\mathcal{H}$ real separable Hilbert space,
  $(\mathcal{H}, \mathcal{B}, i)$ abstract Wiener space[3] built on $\mathcal{H}$

$$\widetilde{\int}_{\mathcal{H}} f(x) e^{\frac{i}{2}\|x\|^2} dx = \int_{\mathcal{B}} f(e^{i\pi/4}x) d\mu(x)$$

[3] L. Gross. Measurable functions on Hilbert spaces. Trans. Am. Math. Soc.105 (3), 372-390 (1962)

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- $\mathcal{H}_t$ Cameron-Martin space,
  $(\mathcal{H}_t, C_t, i)$ Classical Wiener space

$$\tilde{\int}_{\mathcal{H}_t} f(\gamma) e^{\frac{i}{2\hbar}\|\gamma\|^2} d\gamma = \int_{C_t} f(e^{i\pi/4}\sqrt{\hbar}\omega) d\mathbb{P}(\omega)$$

[3] L. Gross. Measurable functions on Hilbert spaces. Trans. Am. Math. Soc.105 (3), 372-390 (1962)

Application to the infinite dimensional oscillatory integral of $f(\gamma) = \int_0^t \mathbf{a}(\gamma(s) + x) \cdot \dot{\gamma}(s) ds$

$P_n : \mathcal{H}_t \to \mathcal{H}_t$, $n \geq 1$ projectors onto the subspaces of piece-wise linear paths.

$$\begin{aligned}
&\lim_{n\to\infty} (2\pi i\hbar t/n)^{-nd/2} \int_{P_n\mathcal{H}}^{o} e^{i\frac{\|\gamma\|^2}{2\hbar}} f(\gamma) d\gamma \\
=&\lim_{n\to\infty} (2\pi t/n)^{-nd/2} \int_{P_n\mathcal{H}}^{o} e^{-\frac{\|\gamma\|^2}{2}} f(e^{i\pi/4}\sqrt{\hbar}\gamma) d\gamma \\
=&\lim_{n\to\infty} \mathbb{E}[e^{i\pi/4}\sqrt{\hbar} \int_0^t \mathbf{a}(e^{i\pi/4}\sqrt{\hbar} B_n(s) + x) \dot{B}_n(s) ds] \\
=&e^{i\pi/4}\sqrt{\hbar} \int_{C_t} \left(\int_0^t \mathbf{a}(e^{i\pi/4}\sqrt{\hbar}\omega(s) + x) \circ d\omega(s) \right) d\mathbb{P}(\omega)
\end{aligned}$$

Theorem

Let $a_i = \hat{\mu}_i$, $i = 1, 2, 3$, $\psi_o \in S(\mathbb{R}^d)$, μ_i *and* $\hat{\psi}_0$ *compactly supported. Let* $g_m^x : \mathcal{H}_t \to \mathbb{C}$ *defined as*

$$g_m^x(\gamma) := \psi_0(\gamma(t) + x)\left(\int_0^t \mathbf{a}(\gamma(s) + x) \cdot \dot{\gamma}(s) ds\right)^m$$

and $P_n : \mathcal{H}_t \to \mathcal{H}_t$, $n \geq 1$ *projectors onto the subspaces of piece-wise linear paths. Then for any* $m \geq 0$,

$$\lim_{n\to\infty}(2\pi i\hbar t/n)^{nd/2}\int_{P_n\mathcal{H}}^{o} e^{i\frac{\|P_n\gamma\|^2}{2\hbar}} g_m^x(P_n\gamma)d(P_n\gamma)$$
$$= \int_{C_t}\left(e^{i\pi/4}\sqrt{\hbar}\int_0^t \mathbf{a}\left(e^{i\pi/4}\sqrt{\hbar}\omega(s) + x\right) \circ d\omega(s)\right)^m$$
$$\psi_0\left(e^{i\pi/4}\sqrt{\hbar}\omega(t) + x\right) d\mathbb{P}(\omega),$$

$\mathbb{P}$ *being Wiener measure on* $(C_t, \mathcal{B}(C_t))$.

The convergence of the Dyson series

$$H\psi = \frac{1}{2}\left(-i\hbar\nabla - \lambda \mathbf{a}(x)\right)^2 \psi, \qquad \psi \in C_0^\infty(\mathbb{R}^3)$$

$$U(t) = e^{-\frac{i}{\hbar}Ht}, \qquad t \in \mathbb{R}$$

Theorem

Let $a_i(x) = \int_{\mathbb{R}^3} e^{ix\cdot k} d\mu_i(k)$, $i = 1, 2, 3$, *and* $\psi_0(x) = \int_{\mathbb{R}^3} e^{ix\cdot k}\hat{\psi}_0(k)dk$, *with* μ_i *and* $\hat{\psi}_0$ *compactly supported. Then there exists a* $\lambda^* > 0$ *such that the Dyson expansion for the vector* $\psi(t) = U(t)\psi_0$

$$\psi(t) = \sum_n \lambda^n \phi_n(t)$$

in convergent in $L^2(\mathbb{R}^d)$ *for* $|\lambda| < \lambda^*$.

Theorem

Under the assumption above the solution of the Schrödinger equation with magnetic field

$$i\hbar\partial_t\psi(t) = H\psi(t,x), \quad \psi(0,x) = \psi_0(x), \quad H = \frac{1}{2}(-i\hbar\nabla - \lambda \mathbf{a}(x))^2$$

can be expressed by the perturbative Dyson series expansion as $e^{-\frac{i}{\hbar}Ht}\psi_0 = \sum_{m=0}^{\infty}\lambda^m\psi_m(t),$, *with* ψ_m *given by*

$$\frac{1}{m!}\left(-\frac{i}{\hbar}\right)^m \widetilde{\int_{\mathcal{H}_t}} \left(\int_0^t \mathbf{a}(\gamma(s)+x)\cdot\dot\gamma(s)ds\right)^m e^{\frac{i}{2\hbar}\int_0^t \|\dot\gamma(s)\|^2 ds}$$

$$\psi_0(\gamma(t)+x)d\gamma$$

$$= \frac{1}{m!}\left(-\frac{i}{\hbar}\right)^m \mathbb{E}\Big[\left(e^{i\pi/4}\sqrt{\hbar}\int_0^t \mathbf{a}\left(e^{i\pi/4}\sqrt{\hbar}\omega(s)+x\right)\circ d\omega(s)\right)^m$$

$$\psi_0\left(e^{i\pi/4}\sqrt{\hbar}\omega(t)+x\right)\Big]$$

The expansion is convergent in $L^2(\mathbb{R}^3)$ *for* $\lambda \in \mathbb{C}$, *with* $|\lambda| < \lambda^*$.

Independence of the approximation procedure

Let $\{P_n\}$ projection operators onto finite dimensional subspaces $P_n(\mathcal{H}) \subset \mathcal{H}$, such that $P_n \to I$ strongly. The infinite dimensional oscillatory integral of $f : \mathcal{H} \to \mathbb{C}$ is defined as

$$\lim_{n\to\infty} (2\pi i\hbar)^{-n/2} \int_{P_n\mathcal{H}}^{o} e^{\frac{i}{2\hbar}\langle P_n x, P_n x\rangle} f(P_n x) dP_n x$$

if the limit exists and *it is independent of the sequence* $\{P_n\}$.

Introduction of counterterms

Theorem

Let $\mathbf{a} : \mathbb{R}^3 \to \mathbb{R}^3$ *be a linear vector field,* $\mathbf{B} = rot(\mathbf{a})$ *and let* $\{e_k\}$ *be an orthonormal basis of* $\mathcal{H}_t$ *and* P_n *be the projector onto the span of* $e_1, \dots, e_n$.. *Then by setting*

$$r_n := \mathbf{B} \cdot \frac{1}{2} \sum_{k=1}^{n} \int_0^t e_k(s) \wedge \dot{e}_k(s) ds$$

the sequence of finite dimensional oscillatory integrals

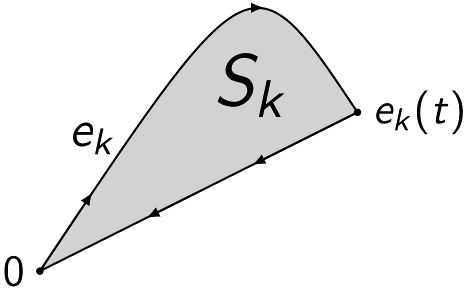
$$(2\pi i\hbar)^{-n/2} \widetilde{\int_{P_n\mathcal{H}}} e^{\frac{i}{2\hbar}\|\gamma_n\|^2} e^{-\frac{i}{\hbar}\left(\int_0^t \mathbf{a}(\gamma_n(s)+x)\cdot\dot{\gamma}_n(s)ds - r_n\right)} \psi_0(\gamma_n(t)+x) d\gamma_n.$$

converges independently of the choice of $\{e_k\}$ *to*

$$\mathbb{E}\left[\psi_0(e^{i\pi/4}\sqrt{\hbar}\omega(t) + x) e^{-\frac{i}{\hbar} e^{i\pi/4}\sqrt{\hbar} \int_0^t \mathbf{a}(e^{i\pi/4}\sqrt{\hbar}\omega(s)+x)\circ d\omega(s)}\right].$$

Introduction of counterterms

$$r_n := \mathbf{B}\cdot\frac{1}{2}\sum_{k=1}^{n}\int_0^t e_k(s)\wedge\dot{e}_k(s)ds = (B_1, B_2, B_3)\cdot\sum_{k=1}^{n}(S_{k,1}, S_{k,2}, S_{k,3})$$



Remark: $r_n = Tr[P_n T]$, with $T : \mathcal{H}_t \to \mathcal{H}_t$ Hilbert Schmidt but not trace-class!

An example of non-essentially self-adjoint Hamiltonian: the inverse quartic oscillator

$$H\psi(x) = -\frac{\hbar^2}{2m}\Delta\psi(x) + V(x)\psi(x), \qquad \psi \in C_0^\infty(\mathbb{R}^d)$$

$$V(x) = \frac{1}{2}x\Omega^2 x + \lambda|x|^4$$

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- If $\lambda > 0$ then H is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$ and $D(H) = D(\Delta) \cap D(|x|^4)$.
- if $\lambda < 0$, as V is real, H commutes with complex conjugation and admits self-adjoint extensions. H is in the "limit circle" case and is not essentially self-adjoint.

How to determine the quantum dynamics?

An example of non-essentially self-adjoint Hamiltonian: the inverse quartic oscillator

[S. M. (2008)]: FPI construction of the quantum dynamics associated to a non essentially self-adjoint Hamiltonian

$$H = -\frac{\hbar^2}{2m}\Delta + \frac{\Omega^2}{2}|x|^2 + \lambda|x|^4, \qquad \lambda < 0$$

- Main result: there exist a strongly continuous contraction semigroup $T(t) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, $t \geq 0$ such that

$$\langle \phi, T(t)\psi_0 \rangle = (i)^{d/2} \int_{\mathbb{R}^d \times C_t} e^{\frac{1}{2\hbar}\int_0^t (\sqrt{\hbar}\omega(s)+x)\Omega^2(\sqrt{\hbar}\omega(s)+x)ds}$$
$$e^{-i\frac{\lambda}{\hbar}\int_0^t |\sqrt{\hbar}\omega(s)+x|^4 ds}\bar{\phi}(e^{i\pi/4}x)\psi_0(e^{i\pi/4}\sqrt{\hbar}\omega(t)+e^{i\pi/4}x)W(d\omega)dx$$

- it solves the Schrödinger equation in a distributional sense:

$$i\hbar\frac{d}{dt}\langle \phi, T(t)\psi_0 \rangle = \langle H\phi, T(t)\psi_0 \rangle,$$

Problem: what kind of dynamic does $T(t)$ describe?

$$H\psi = -\frac{\hbar^2}{2m}\Delta\psi + \frac{\Omega^2}{2}|x|^2\psi + \lambda|x|^4\psi, \qquad \lambda < 0, \psi \in C_0^\infty(\mathbb{R}^d)$$

Different self-adjoint extensions $H_\alpha = H_\alpha^\dagger$, $\alpha \in A$

$$\downarrow$$

Different quantum evolutions $U_\alpha(t) = e^{-\frac{i}{\hbar}H_\alpha t}$, , $\alpha \in A$

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Conjecture: Feynman evolution is a convex superposition of U_α, $\alpha \in A$

$$T(t) = \int_A U_\alpha(t) d\mu(\alpha)$$

(for $V(x) = -\frac{\beta}{|x|^2}$, $\frac{\beta m}{\hbar^2} > \frac{1}{4}$, see Nelson (1964) and Radin (1975))

A (still open) problem with (preliminary) ambiguous solutions: FPIs on Riemannian manifolds

- (M,g) d-dimensional Riemannian manifold, classical Hamiltonian for a free particle $H(q,p) = g^{ij}(q)p_i p_j$
- Feynman path integral:

$$\psi(t,x) = \int e^{\frac{i}{\hbar}S_t(\gamma)} e^{-i\hbar K \int_0^t R(\gamma(s))ds} \psi_0(\gamma(0)) d\gamma \qquad (4)$$

$$K = \frac{1}{6}, \qquad K = \frac{1}{12}, \qquad K = \frac{1}{3}, \qquad K = 0, \qquad K = ????$$

- The ambiguity on the value of K is the counterpart of the ambiguity in the quantization of the classical Hamiltonian

$$H(q,p) = g^{ij}(q)p_i p_j \quad \mapsto \quad H = -\frac{\hbar^2}{2}\Delta - \hbar^2 KR$$

Thanks!