Entropy dissiation for some Lindblad equations descibing many-body kinetic systems

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Quantum Markov Semigroups

Let $\mathcal{H}$ be a finite dimensional Hilbert space. Make $\mathcal{B}(\mathcal{H})$ a Hilbert space by equipping it with the Hilbert-Schmidt inner product

$$\langle A, B \rangle = \mathrm{Tr}[A^* B].$$

A linear transformation $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ has an adjoint with respect to this inner product that we denote by $\Phi^\dagger$. A linear transformation $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is *unital* in case $\Phi(I) = I$ and it is *trace preserving* (TP) in case for all $A \in \mathcal{B}(\mathcal{H})$, $\mathrm{Tr}[\Phi(A)] = \mathrm{Tr}[A]$. Φ is unital if and only if $\Phi^\dagger$ is trace preserving.

For A, B, C, D in $\mathcal{B}(\mathcal{H})$, consider the block operator

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = A \otimes E_{1,1} + B \otimes E_{1,2} + C \otimes E_{2,1} + D \otimes E_{2,2}$$

which we may think of as an operator on $\mathcal{H} \oplus \mathcal{H}$, or equivalently $\mathcal{H} \otimes \mathbb{C}^2$. Then Φ is 2-*positive* in case

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \geq 0 \Rightarrow \begin{bmatrix} \Phi(A) & \Phi(B) \\ \Phi(C) & \Phi(D) \end{bmatrix} \geq 0 \ ;$$

i.e., in case $\Phi \otimes I_{M_2(\mathbb{C})}$ is positivity preserving. The definition of n-positivity is made in the obvious way, and then Φ is *completely positive* (CP) in case Φ is n-positive for all $n \in \mathbb{N}$.

We will be particularly concerned with completely positive maps Φ that satisfy $\Phi^\dagger = \Phi$ that are unital and hence trace preserving. CPTP maps have a particularly simple *Stinespring factorization*:
Define $\Psi_m : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \otimes M_m(\mathbb{C})$ by

$$\Psi_m(A) = \begin{bmatrix} A & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix} = A \otimes I_{M_m(\mathbb{C})} \ .$$

Then for some unitary U on $\mathcal{H} \otimes M_m(\mathbb{C})$, and some m,

$$\begin{aligned} \Phi(A) &= \mathrm{Tr}_2(U^* \Psi_m(A) U) \\ &= \sum_{j=1}^m U_{1,j}^* A U_{1,j} = \sum_{j=1}^m V_j^* A V_j \ . \end{aligned}$$

A *Quantum Markov Semigroup* (QMS) on $\mathcal{B}(\mathcal{H})$ is a semigroup $\{P_t\}_{t\geq 0}$ on $\mathcal{B}(\mathcal{H})$ such that each P_t is unital and CP.
Let Φ be any CP map on $\mathcal{B}(\mathcal{H})$. Then $\{e^{t\Phi}\}_{t\geq o}$ is a CP semigroup. For any $G \in \mathcal{B}(\mathcal{H})$, $A \mapsto e^{tG^*}Ae^{tG}$ gives another example. Now define

$$L(A) := \Phi(A) - (G^*A + AG) \ .$$

By the Trotter product formula, $\{e^{tL}\}_{t\geq 0}$ is a CP semigroup. It is unital if and only if $L(I) = 0$, and hence

$$\Phi(I) = G^* + G \quad \text{so that} \quad G = \frac{1}{2}\Phi(I) + iH \ .$$

Hence

$$L(A) = \phi(A) - \frac{1}{2}(\Phi(I)A + A\Phi(I)) + i[H, A] \ .$$

Lindblad's Theorem says that every QMS is if the form $\{e^{tL}\}_{t\geq 0}$ where for some CP map Φ and some self-adjoint $H \in \mathcal{B}(\mathcal{H})$,

$$L(A) = \Phi(A) - \frac{1}{2}(\Phi(I)A + A\Phi(I)) + i[H, A] \ .$$

With $\Phi(A) = \sum_{j=1}^m V_j^* A V_j$,

$$L(X) = \sum_{j=1}^m \left(V_j^* X V_j - \frac{1}{2} V_j^* V_j X - \frac{1}{2} X V_j^* V_j \right) + i[H, X]$$

If $L = L^\dagger$ and $H = 0$,

$$L = -\frac{1}{2} \sum_{j=1}^n [V_j, [V_j^*, X]] = -\frac{1}{2} \sum_{j=1}^n [V_j^*, [V_j, X]] \ .$$

Then $\dfrac{\partial}{\partial t}\rho(t) = L\rho(t)$ is a sort of "quantum heat equation".

Relative entropy

Given two density matrices $\rho, \sigma \in \mathcal{B}(\mathcal{H})$, the relative entropy of ρ with respect to σ, $D(\rho||\sigma)$, is defined by

$$D(\rho||\sigma) := \mathrm{Tr}[\rho(\log \rho - \log \sigma)]$$

Now suppose that $\{P_t\}_{t\geq 0}$ is a QMS and for each t, $P_t = P_t^\dagger$ so that P_t takes density matrices to density matrices. Suppose also that $P_t\sigma = \sigma$. By the Data Processing Inequality (DPI),

$$D(P_t\rho||\sigma) = D(P_t\rho||P_t\sigma) \leq D(\rho||\sigma) \ .$$

The relative entropy dissipation under P_t is

$$I(\rho||\sigma) := -\frac{\mathrm{d}}{\mathrm{d}t} D(P_t^\dagger \rho||\sigma)\Big|_{t=0} \geq 0 \ .$$

Entropy-Entropy Dissipation

A *generalized logarithmic Sobolev inequality*) (GLSI) is an inequality of the form

$$I(\rho||\sigma) \geq cD(\rho||\sigma)$$

for $c > 0$. Such an inequality yields

$$D(P_t\rho||P_t\sigma) \leq e^{-ct}D(\rho||\sigma) \ .$$

Suppose such an inequality holds with $\sigma = \frac{1}{\dim(\mathcal{H})}I$. Let h be self adjoint with $\mathrm{Tr}[h] = 0$. Then for all ϵ small enough, $\rho_\epsilon := \sigma + \epsilon h$ is a density matrix, and so for all such ϵ,

$$I(\rho_\epsilon||\sigma) \geq cD(\rho_\epsilon||\sigma) \ .$$

Expanding to second order in ϵ yields the *spectral gap inequality* (SGI)

$$-\langle h, Lh\rangle \geq \frac{c}{2}\langle h, h\rangle \quad \text{for} \quad \langle h, I\rangle = 0 \ .$$

As we have seen, a spectral gap inequality is weaker than a generalized logarithmic Sobolev inequality. There is also a stronger inequality, namely a *logarithmic Sobolev inequality*.

Theorem

Let L be a quantum Markov semigroup generator with $L = L^\dagger$, and let σ satisfy $L\sigma = 0$. Then for any density matrix ρ.

$$I(\rho||\sigma) \geq 4 \sum_{j=1}^{n} \mathrm{Tr}\,|[V_j, \sqrt{\rho}]|^2 \ .$$

A *logarithmic Sobolev inequality* (LSI) for the QMS generators considered here is an inequality of the form

$$\sum_{j=1}^{n} \mathrm{Tr}\,|[V_j, \sqrt{\rho}]|^2 \geq \frac{c}{4} D(\rho||\sigma) \ .$$

By the theorem, this implies

$$I(\rho||\sigma) \geq cD(\rho||\sigma) \ ,$$

which in turn implies

$$-\langle h, Lh\rangle \geq \frac{c}{2}\langle h, h\rangle \ .$$

It is known that

$$D(\rho||\sigma) \geq \frac{1}{2}\|\rho - \sigma\|_1 = \frac{1}{2}\,\mathrm{Tr}[|\rho - \sigma|] \ .$$

An easy computation yields (since $L = L^\dagger$)

$$I(\rho||\sigma) = -\operatorname{Tr}[(L\rho)(\log\rho - \log\sigma)] \ .$$

Using the integral representation

$$\log X = \int_0^\infty \left(\frac{1}{\lambda+1} - \frac{1}{\lambda+X} \right) \mathrm{d}\lambda \ ,$$

one deduces

$$I(\rho||\sigma) = \frac{1}{2} \sum_j \int_0^\infty \operatorname{Tr}\left[[V_j,\rho]^* \frac{1}{\lambda+\rho} [V_j,\rho] \frac{1}{\lambda+\rho} \right] \mathrm{d}\lambda \ .$$

The functional

$$(K,X) \mapsto \int_0^\infty \mathrm{Tr}\left[K^* \frac{1}{\lambda+X} K \frac{1}{\lambda+X}\right] \mathrm{d}\lambda$$

is jointly convex on $M_n(\mathbb{C}) \times M_n^{++}(\mathbb{C})$ by a Theorem of Lieb from 1973. Just as the joint convexity of the relative entropy leads to the DPI, this functional is also monotone decreasing under

$$(K,X) \mapsto (\Phi(K), \Phi(X))$$

for any CPTP map, a result due to Petz in 1996.
This is fundamentally important in a geometric approach to proving GLSI that has been developed by myself and Jan Maas, and other researchers as well – it is a very active field. But it is unknown how to check the geometric conditions in the cases we discuss here. Instead, we shall directly prove an LSI.

The N-particle binary collision model

Consider a d-dimensional Hilbert space $\mathcal{H}$ and a single particle Hamiltonian h with d distinct eigenvalues $e_0, \dots, e_{d-1}$. Let $\{\psi_0, \dots, \psi_{d-1}\}$ be an orthonormal basis for $\mathcal{H}$ with $h\psi_j = e_j\psi_j$ for all $0 \le j \le d-1$. The corresponding N-particle Hamiltonian on $\mathcal{H}_N = \otimes^N\mathcal{H}$ is given by

$$H_N = \sum_{j=1}^{N} I \otimes \cdots \otimes h \otimes \cdots \otimes I$$

where h is in the jth position. The eigenvalues of H_N are indexed by the multindices $\alpha \in \{0, \dots, d-1\}^N$, and are given by

$$e(\alpha) = e_{\alpha_1} + \cdots e_{\alpha_N} \quad \text{where} \quad \alpha_j \in \{0, \dots, d-1\}\ ,\ j = 1, \dots, N\ .$$

Defining

$$\Psi_\alpha := \psi_{\alpha_1} \otimes \cdots \otimes \psi_{\alpha_N} \ ,$$

$\{\Psi_\alpha \ : \ \alpha \in \{0, \dots d-1\}^N\}$ is an orthonormal basis of $\mathcal{H}_N$ consisting of eigenvectors of H_N. For a multiindex α, and $k \in \{0, \dots, d-1\}$ define

$$k_m(\alpha) = \#\{1 \leq j \leq N \ \ \alpha_j = m \ \}$$

where for a set A, $\#A$ denotes the cardinality of A. Thus, a second expression for $e(\alpha)$ is

$$e(\alpha) = \sum_{m=0}^{d-1} k_m(\alpha) e_k \ .$$

We consider energy conserving binary collisions: If before a collision the state of the system is $|\Psi_\gamma\rangle\langle\Psi_\gamma|$, then after the collision it given by a density matrix ρ all of whose eigenstates are linear combinations of vectors the form Ψ_δ where for some $i < j$, $\delta_\ell = \gamma_\ell$ for $\ell \notin \{i,j\}$, and

$$e_{\gamma_i} + e_{\gamma_i} = e_{\delta_i} + e_{\delta_i}$$

Suppose that the spectrum of h is such that the spectrum of H_2 on $\mathcal{H} \otimes_{\rm sym} \mathcal{H}$ is non-degenerate. Then for any $0 \leq m_1, m_2, m_3, m_4 \leq d-1$,

$$e_{m_1} + e_{m_2} = e_{m_3} + e_{m_4} \iff \{m_1, m_2\} = \{m_3, m_4\} \ .$$

Suppose further that the pair of equations

$$\sum_{m=1}^{d-1} k_m e_n = E \quad \text{and} \quad \sum_{m=1}^{d-1} k_m \ = \ N$$

has exactly one solution for each E in the spectrum of $\mathcal{H}_N$. Then γ and δ can only differ differ by a pair transposition. (This is generic.)

The collision rules induce a graph structure on $\mathcal{V}_N := \{0, \dots d-1\}^N$: We say that $\gamma, \delta \in \mathcal{V}_N$ are *adjacent* if they differ by a pair transposition $\pi_{i,j}$ for some $i < j$. If γ is adjacent δ, we write $[\gamma, \delta]$ to denote the corresponding edge in $\mathcal{E}_N$. Note that $[\gamma, \delta] = [\delta, \gamma]$. We denote this graph by $\mathcal{G}_N$.

The connected components of $\mathcal{G}_N$ are indexed by the $\mathbf{k} = (k_0, \dots, k_{d-1})$ such that $\sum_{m=1}^{d-1} k_m = N$, and the corresponding vertex set, $\mathcal{V}_{N,\mathbf{k}}$ consists of all α such that $k_j(\alpha) = k_j$ for each $j = 0, \dots, d-1$. Evidently, the cardinality of $\mathcal{V}_{N,\mathbf{k}}$ is

$$d_{\mathbf{k}} := \frac{N!}{k_0! \cdots k_{r-1}!} ,$$

and the valency $v(\alpha)$ of the vertex α:

$$v(\alpha) = \sum_{m<n} k_m(\alpha) k_n(\alpha) .$$

Definition

For $\mathbf{k} = (k_0, \ldots, k_{d-1}) \in (\mathbb{Z}_{\geq 0})^d$ with $\sum_{m=0}^{d-1} k_m = N$,

$$\kappa_{\min} := \min\{k_0, \ldots, k_{d-1}\}$$

and

$$r(\mathbf{k}) := \frac{\kappa_{\min}}{N}$$

It follows directly from the definition that for all $\alpha \in \mathcal{V}_{N,\mathbf{k}}$,

$$v(\alpha) \geq r^2(\mathbf{k}) N^2 \ .$$

We write $\mathcal{H}_{N,\mathbf{k}}$ to denote the corresponding eigenspaces of H_N, and we write $P_{N,\mathbf{k}}$ to denote the orthogonal projection onto $\mathcal{H}_{N,\mathbf{k}}$ in $\mathcal{H}_N$.

The Lindblad equations

For each pair of multindices α and β, define

$$F_{\alpha\beta} := |\Psi_\alpha\rangle\langle\Psi_\beta| \qquad \text{and} \qquad L_{\alpha\beta} = F_{\alpha\beta} - F^*_{\alpha\beta} = F_{\alpha\beta} - F_{\beta\alpha} \ .$$

$$\mathcal{L}(X) = \frac{1}{N-1}\sum_{[\alpha,\beta]\in\mathcal{E}_N} [L_{\alpha\beta},[L_{\alpha\beta},X]] = -\frac{1}{N-1}\sum_{[\alpha,\beta]\in\mathcal{E}_N} [L^*_{\alpha\beta},[L_{\alpha\beta},X]] \ .$$

and note that $\mathcal{L} = \mathcal{L}^\dagger$.
Since for each α,β with $[\alpha,\beta]\in\mathcal{E}_N$, $P_{N,\mathbf{k}}L_{\alpha\beta} = L_{\alpha\beta}P_{N,\mathbf{k}}$, it follows that for all X

$$P_{N,\mathbf{k}}\mathcal{L}(X)P_{N,\mathbf{k}} = \mathcal{L}(P_{N,\mathbf{k}}XP_{N,\mathbf{k}}) \ .$$

Define

$$\mathcal{B}_\mathbf{k} := \{ \ X \in \mathcal{B}(\mathcal{H}_N) \ : \ X = P_{N,\mathbf{k}}XP_{N,\mathbf{k}} \ \} \ ,$$

which is a sub-algebra of $\mathcal{B}(\mathcal{H}_N)$. Each $\mathcal{B}_\mathbf{k}$ is invariant under $\mathcal{L}$ and the corresponding Quantum Markov semigroup $\mathcal{P}_t = e^{t\mathcal{L}}$.

Note that for $\alpha \neq \beta$, $L_{\alpha\beta}^2 = -F_{\alpha,\alpha} - F_{\beta,\beta}$. Since

$$[L_{\alpha\beta}, [L_{\alpha\beta}, X]] = L_{\alpha\beta}^2 X + X L_{\alpha\beta}^2 - 2 L_{\alpha\beta} X L_{\alpha\beta} \ ,$$

For $X \in \mathcal{B}_{\mathbf{k}}$,

$$\mathcal{L}(X) = \frac{2}{N-1} \sum_{[\alpha,\beta] \in \mathcal{E}_N} L_{\alpha\beta}^* X L_{\alpha\beta} - 2 \frac{v(\mathbf{k})}{N-1} X \ .$$

Defining $\Phi_{\mathbf{k}}(X) := \frac{1}{2v(\mathbf{k})} \sum_{[\alpha\beta] \in \mathcal{E}_{N,\mathbf{k}}} L_{\alpha\beta}^* X L_{\alpha\beta}$ so that $\Phi_{\mathbf{k}}(I) = I$,

$$e^{t\mathcal{L}}\Big|_{\mathcal{B}_{\mathbf{k}}} = e^{-2tv(\mathbf{k}/(N-1))} \sum_{\ell=0}^{\infty} \frac{(t2v(\mathbf{k})/(N-1))^\ell}{\ell!} \Phi_{\mathbf{k}}^\ell$$

For any α and any pair permutation $\pi_{i,j}$ of $\{1,\ldots,N\}$, either $[\alpha,\pi_{i,j}(\alpha)]$ is an edge or else $\alpha=\pi_{i,j}(\alpha)$, and each β with $[\alpha,\beta]\in\mathcal{E}_N$ satisfies $\beta=\pi_{i,j}(\alpha)$ for exactly one pair $i<j$. Therefore

$$\sum_{[\alpha\beta]\in\mathcal{E}_{N,\mathbf{k}}} L^*_{\alpha\beta}XL_{\alpha\beta} = \sum_{i<j}\sum_{\alpha\in\mathcal{V}_{N,\mathbf{k}}} L^*_{\alpha,\pi_{i,j}(\alpha)}XL_{\alpha,\pi_{i,j}(\alpha)}\ .$$

Hence if we define

$$\Phi_{\mathbf{k},(i,j)}(X) := \binom{N}{2}\frac{1}{2v(\mathbf{k})}\sum_{\alpha\in\mathcal{V}_{N,\mathbf{k}}} L^*_{\alpha,\pi_{i,j}(\alpha)}XL_{\alpha,\pi_{i,j}(\alpha)}\ ,$$

$$\Phi_{\mathbf{k}} = \binom{N}{2}^{-1}\sum_{i<j}\Phi_{\mathbf{k},(i,j)}$$

The spectrum of $\mathcal{L}$

Lemma

For $\alpha \neq \beta$ and any γ, δ,

$$
\begin{aligned}
[L_{\alpha\beta}, [L_{\alpha\beta}, F_{\gamma\delta}]] &= 2\delta_{\beta\gamma}\delta_{\beta\delta}F_{\alpha\alpha} + 2\delta_{\alpha\gamma}\delta_{\alpha\delta}F_{\beta\beta} \\
&- 2\delta_{\beta\gamma}\delta_{\alpha\delta}F_{\alpha\beta} - 2\delta_{\alpha\gamma}\delta_{\beta\delta}F_{\beta\alpha} \\
&- (\delta_{\beta\gamma} + \delta_{\alpha\gamma} + \delta_{\beta\delta} + \delta_{\alpha\delta})F_{\gamma\delta} \ .
\end{aligned}
$$

The next theorem provides a basis of eigenvectors of $\mathcal{L}$ partly in terms of $\Delta_{\mathcal{G}(N)}$, the classical graph Laplacian associated to $\mathcal{G}(N)$. Recall that

$$
\Delta_{\mathcal{G}(N)} f(\gamma) = \left(\sum_{\{\delta \ : \ [\delta,\gamma] \in \mathcal{E}(N)\}} f(\delta) \right) - v(\gamma) f(\gamma) \ .
$$

Theorem

For $\gamma \neq \delta$, the operators $S_{\gamma\delta} := F_{\gamma\delta} + F_{\delta\gamma}$ and $L_{\gamma\delta}$ are eigenvectors of $-(N-1)\mathcal{L}$, each with one of three eigenvalues

$$v(\gamma) + v(\delta) - 2 \ , \quad v(\gamma) + v(\delta) \quad \text{or} \quad v(\gamma) + v(\delta) + 2 \ .$$

Finally,

$$(N-1)\mathcal{L}(F_{\gamma\gamma}) = 2 \sum_{\delta \in \mathcal{G}_N} [\Delta_{\mathcal{G}_N}]_{\gamma\delta} F_{\delta\delta}$$

where $\Delta_{\mathcal{G}(N)}$ is the graph Laplacian associated with $\mathcal{G}(N)$. If $\vec{u}$ is an eigenvector of $\Delta_{\mathcal{G}(N)}$ with eigenvalue Λ, then $X_u := \sum_{\delta \in \mathcal{G}(N)} u_\delta F_{\delta\delta}$ then $\mathcal{L}(X_u) = \Lambda X_u$.

Theorem

Let $\mathbf{k}$ *be such that* $k_m > 0$ *for at least two values of* m*. Then, for all such* $\mathbf{k}$*, every eigenvalue* λ *of* $-\Delta_{\mathcal{G}_{N,\mathbf{k}}}$ *satisfies*

$$N \leq \lambda \leq N(N-1) \ . \tag{1}$$

The lower bound is sharp, in that N *is the spectral gap for* $\Delta_{\mathcal{G}_{N,\mathbf{k}}}$ *for all such* $\mathbf{k}$*. The upper bound is also sharp, in that for* $\mathbf{k} = (1, 1, \ldots 1)$*, so that* $\Delta_{\mathcal{G}_{N,\mathbf{k}}}$ *generates the random transposition walk on the symmetric group on* N *letters,* $N(N-1)$ *is the largest eigenvalue of* $-\Delta_{\mathcal{G}_{N,\mathbf{k}}}$

Thus whenever $r(\mathbf{k}) \geq r > 0$, so that $v(\mathbf{k}) \geq \binom{d}{2} r^2 N^2$,

$$N \geq \frac{4}{d(d-1)r^2} \Rightarrow 2(v(\mathbf{k}) - 1)/(N-1) > 2N/(N-1) \ ,$$

and the spectral gap of $\mathcal{L}$ coincides with that of $\frac{2}{N-1}\Delta_{\mathcal{G}_{N,\mathbf{k}}}$.

Theorem

Let $\mathbf{k}$ be such that $r(\mathbf{k}) \geq r > 0$. Then for all $N > 4/(d(d-1)r^2)$, the spectral gap of $\mathcal{L}$ on $\mathcal{B}_{\mathbf{k}}$ is $2N/(N-1)$.

The invariant subspace $\mathcal{B}_{\mathbf{k}}$ to which ρ belongs splits into two further invariant subspaces, one "quantum" and the other "classical":

$$\mathcal{B}_{\mathbf{k}} = \mathcal{Q}_{\mathbf{k}} \oplus \mathcal{C}_{\mathbf{k}}$$

where

$$\mathcal{Q}_{\mathbf{k}} = \mathrm{span}(\{F_{\alpha,\beta} \ : \ \alpha, \beta \in \mathcal{V}_{N,\mathbf{k}} \ , \ \alpha \neq \beta \ \}$$

and

$$\mathcal{C}_{\mathbf{k}} = \mathrm{span}(\{F_{\alpha,\alpha} \ : \ \alpha \in \mathcal{V}_{N,\mathbf{k}} \ \} \ .$$

When $r(\mathbf{k}) \geq r > 0$, for large N, the least eigenvalue of $-\mathcal{L}\big|_{\mathcal{Q}_{\mathbf{k}}}$ is of the same order – $\mathcal{O}(N)$ – as the largest eigenvalue of $-\mathcal{L}\big|_{\mathcal{C}_{\mathbf{k}}}$, while the spectral gap is $2N(N-1)$. The "quantum" degrees of freedom get washed out very quickly, leaving one with a classical Markov process, as we now explain.

For each $\mathbf{k}$, define

$$\sigma_{\mathbf{k}} := \frac{1}{d_{\mathbf{k}}} P_{N,\mathbf{k}} \ . \tag{2}$$

The density matrix $\sigma_{\mathbf{k}}$ spans the null space of $\mathcal{L}$, and all of non-zero eigenvalues of $\mathcal{L}$ are negative. Therefore, if ρ is any density matrix on $\mathcal{H}_N$ such that $\rho \in \mathcal{B}_{\mathbf{k}}$,

$$\lim_{t\to\infty} \mathcal{P}_t \rho = \sigma_{\mathbf{k}} \ . \tag{3}$$

We are interested in measuring the rate of convergence of $\mathcal{P}_t\rho$ to $\sigma_{\mathbf{k}}$. We have from the spectral gap computation that when $r(\mathbf{k}) \geq r > 0$ and N is sufficiently large,

$$\mathrm{Tr}[(\mathcal{P}_t\rho - \sigma_{\mathbf{k}})^2] \leq e^{-4t}\, \mathrm{Tr}[(\rho - \sigma_{\mathbf{k}})^2] \ . \tag{4}$$

This may be sharpened.

Since $\sigma_{\mathbf{k}} \in \mathcal{C}_{\mathbf{k}}$,

$$\begin{aligned} \mathrm{Tr}[(\mathcal{P}_t\rho - \sigma_{\mathbf{k}})^2] &= \mathrm{Tr}[(\mathcal{P}_t(\rho_{\mathcal{Q}}))^2] + \mathrm{Tr}[(\mathcal{P}_t(\rho_{\mathcal{C}}) - \sigma_{\mathbf{k}})^2] \\ &\leq e^{-t2(v(\mathbf{k})-1)/(N-1)}\,\mathrm{Tr}[\rho_{\mathcal{Q}}^2] + \mathrm{Tr}[(\mathcal{P}_t(\rho_{\mathcal{C}}) - \sigma_{\mathbf{k}})^2]\ . \end{aligned}$$

For $\mathbf{k}$ such that $v(\mathbf{k}) \geq r > 0$, the “quantum” component of ρ, $\rho_{\mathcal{Q}}$ gets washed out very quickly and after a short initial layer, one has

$$\mathcal{P}_t\rho \approx \mathcal{P}_t\rho_{\mathcal{C}}$$

to an extremely accurate degree of approximation for large N. .

The classical Markov semigroup

$\mathcal{C}_{\mathbf{k}}$ is a commutative algebra. We identify a function f on $\mathcal{V}_{N,\mathbf{k}}$ with the operator X_f given by

$$X_f := \sum_{\alpha \in \mathcal{V}_{N,\mathbf{k}}} f(\alpha) F_{\alpha,\alpha} \ ,$$

the map is bijective.

Moreover

$$e^{t\mathcal{L}} X_f = X_{f(t)} \quad \text{where} \quad f(t) := e^{t \frac{2}{N-1} \Delta_{\mathcal{G}_{N,\mathbf{k}}}} f \ . \tag{5}$$

That is, every question about $e^{t\mathcal{L}}\big|_{\mathcal{C}_{\mathbf{k}}}$ reduces to a question about the classical Markov semigroup $e^{t \frac{2}{N-1} \Delta_{\mathcal{G}_{N,\mathbf{k}}}} f$. Because of the relatively rapid decay of $e^{t\mathcal{L}}\big|_{\mathcal{Q}_{\mathbf{k}}}$, one might hope this is also true to some extent for $e^{t\mathcal{L}}\big|_{\mathcal{B}_{\mathbf{k}}}$. This is the case.

Hypercontractivity for $e^{t\mathcal{L}}$

The following norms will be useful: For $1 \leq p < \infty$ and $X \in \mathcal{B}_{N,\mathbf{k}}$ we define

$$\|X\|_{p,\mathbf{k}} := \left(\mathrm{Tr}[\sigma_{\mathbf{k}}(X^*X)^{p/2}]\right)^{1/p} = \left(\frac{1}{d_{\mathbf{k}}}\right)^{1/p} \|X\|_p$$

where $\|X\|_p$ denotes the usual Schatten p-norm of X. For all $p > 2$, and all $X \in \mathcal{B}_{\mathbf{k}}$,

$$\|X\|_{p,\mathbf{k}} = \left(\frac{1}{d_{\mathbf{k}}}\right)^{1/p} \|X\|_p \leq \left(\frac{1}{d_{\mathbf{k}}}\right)^{1/p} \|X\|_2 = d_{\mathbf{k}}{}^{1/2-1/p}\|X\|_{2,\mathbf{k}} \ .$$

For all $p \geq 1$,

$$\|X_f\|_{p,\mathbf{k}} = \|f\|_p$$

where the L^p norm on the right is computed with respect ot the normalized probability measure on $\mathcal{V}_{N,\mathbf{k}}$.

Therefore,

$$\|\mathcal{P}_t X_f\|_{p,\mathbf{k}} = \|e^{t(2/(N-1))\Delta_{\mathcal{G}_{N,\mathbf{k}}}} f\|_p \ .$$

Salez has proved that the semigroup $e^{t(2/(N-1))\Delta_{\mathcal{G}_{N,\mathbf{k}}}}$ is hypercontractive with a log-Sobolev constant that is proportional to $r(\mathbf{k})$. In particular, there is a time $T_\mathbf{k}$ bounded by a universal constant times $\frac{1}{r(\mathbf{k})}$ such that

$$\|\exp(T_\mathbf{k} 2\Delta_{\mathcal{G}_{N,\mathbf{k}}}) f\|_4 \leq \|f\|_2 \ .$$

It follows that for all $X \in \mathcal{C}_\mathbf{k}$ $\|\mathcal{P}_{T_\mathbf{k}} X\|_{4,\mathbf{k}} \leq \|X\|_{2,\mathbf{k}}$. This extends to the whole system:

Theorem

Let $\mathbf{k}$ *be such that* $r(\mathbf{k}) \geq r > 0$. *Then there is a time* $T'_\mathbf{k}$ *bounded by a universal constant times* $1/r$ *such that for all* $Z \in \mathcal{B}_\mathbf{k}$,

$$\|\mathcal{P}_{T'_\mathbf{k}} Z\|_{4,\mathbf{k}} \leq \|Z\|_{2,\mathbf{k}} \ .$$

First, we take care of the quantum sector:

Lemma

Let $\mathbf{k}$ *be such that* $\kappa_{\min} \geq rN$ *where* $r > 0$*. Then for all* $N \geq 2/r$*, and all*

$$t \geq \frac{(\log d + r \log 3)}{4r^2} \ , \tag{6}$$

and all $X \in \mathcal{Q}_{\mathbf{k}}$*,*

$$\|\mathcal{P}_t X\|_{4,\mathbf{k}} \leq \|X\|_{2,\mathbf{k}} \ .$$

Write $X = X_1 + X_2 + X_3$ where the X_j are the components of X in the the three eigenspaces of $\mathcal{L}$ on $\mathcal{Q}_{\mathbf{k}}$.
For large N,

$$\|\mathcal{P}_t X\|_{4,\mathbf{k}} \leq e^{-tr^2 N} \left(\|X_1\|_{4,\mathbf{k}} + \|X_2\|_{4,\mathbf{k}} + \|X_3\|_{4,\mathbf{k}} \right) \ .$$

Then since $\|X\|_{4,\mathbf{k}} \leq {d_{\mathbf{k}}}^{1/4} \|X\|_{2,\mathbf{k}}$ and $d_{\mathbf{k}} \leq d^N$,

$$\begin{aligned} \|\mathcal{P}_t X\|_{4,\mathbf{k}} &\leq e^{-tr^2 N} d_{\mathbf{k}}^{1/4} \left(\|X_1\|_{2,\mathbf{k}} + \|X_2\|_{2,\mathbf{k}} + \|X_3\|_{2,\mathbf{k}} \right) \\ &\leq e^{-tr^2 N + N \log d/4} \left(\|X_1\|_{2,\mathbf{k}} + \|X_2\|_{2,\mathbf{k}} + \|X_3\|_{2,\mathbf{k}} \right) \\ &\leq \sqrt{3} e^{-tr^2 N + N \log d/4} \|X\|_{2,\mathbf{k}} \ . \end{aligned}$$

For $Z \in \mathcal{B}_{\mathbf{k}}$, write $Z = X + Y$, $X \in \mathcal{C}_{\mathbf{k}}$ and $Y \in \mathcal{Q}_{\mathbf{k}}$ Then for $t \geq S_{\mathbf{k}}$, where

$$S_{\mathbf{k}} := \max\{ T_{\mathbf{k}} \ , \ (\log d/4 + \log(3/2)/r^2 \ \} \ ,$$

$$\|\mathcal{P}_t Z\|_{4,\mathbf{k}} \leq \|X\|_{4,\mathbf{k}} + \|Y\|_{4,\mathbf{k}} \leq \sqrt{2}\|Z\|_{2,\mathbf{k}} \ .$$

Lemma (Glimm's Lemma)

Let $P_t = e^{tL}$ be an ergodic Quantum Markov Semigroup with $P_t = P_t^\dagger$ for each t. Suppose that for some $0 < T, C < \infty$

$$\|P_T X\|_4 \leq C\|X\|_2$$

for all X. Suppose also that L has a spectral gap $\lambda > 0$ so that for all Y with $\mathrm{Tr}[Y] = 0$, $\|P_s Y\|_2 \leq e^{-\lambda s}\|Y\|_2$. Then for all X,

$$t \geq \frac{\log(\sqrt{2}C)}{\lambda} \rightarrow \|P_{T+t}X\|_4 \leq \|X\|_2 \ .$$

The proof in the quantum case uses:

Lemma (Carlen and Lieb)

Let $\Phi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ be a completely positive trace preserving map. Then for all X and all $1 \leq p < \infty$,

$$\|\Phi(X)\|_p \leq \|\Phi(|X|)\|_p^{1/2} \|\Phi(|X^*|)\|_p^{1/2}$$

Let $X > 0$ and define $\langle X \rangle := \mathrm{Tr}[X]$, and $Y := X - \langle X \rangle$. Then

$$\begin{aligned}
(P_{t+T}X)^4 &= (P_{t+T}Y + \langle X \rangle)^4 \\
&= (P_T P_t Y)^4 + 4(P_T P_t Y)^3 \langle X \rangle \\
&+ 6(P_{t+T}Y)^2 \langle X \rangle^2 + 4(P_{t+T}Y)\langle X \rangle^3 + \langle X \rangle^4 \ .
\end{aligned}$$

$$\|X\|_2^4 = (\|Y\|_2^2 + \langle X \rangle^2)^2 = \|Y\|_2^4 + 2\|Y\|_2^2 \langle X \rangle^2 + \langle X \rangle^4 \ .$$

We now follow another classical argument of Glimm and apply apply the Stein-Riesz-Thorin Interpolation Theorem.

$$\begin{aligned} \|e^{TL}\|_{2\to 4} &= 1 \\ \|I\|_{2\to 2} &= 1 \ . \end{aligned}$$

Then for $2 < p < 4$, with $\theta(p)$ defined by

$$\frac{1}{p} = (1-\theta)\frac{1}{2} + \theta\frac{1}{4} \ ,$$

so that $\theta(p) = (2p-4)/p$, we have for all Z

$$\|P_{\theta(p)T}Z\|_p \leq \|Z\|_2 \ .$$

Differentiating in p at $p = 2$ yields our LSI:

Theorem

For all $\mathbf{k}$ *such that* $r(\mathbf{k}) \geq r > 0$, *and all* $Z \in \mathcal{B}_{\mathbf{k}}$, *with* $\mathrm{Tr}[\sigma_{\mathbf{k}}|Z|^2] = 1$,

$$\mathrm{Tr}[\sigma_{\mathbf{k}}|Z|^2 \log|Z|^2] \leq 4C_{\mathbf{k}}\mathcal{D}_N(Z, Z) \ ,$$

where the Dirichlet form $\mathcal{D}_N$ *is defined by*

$$\mathcal{D}_N(Z, Z) := -\mathrm{Tr}\,\sigma_{\mathbf{k}}[Z^*\mathcal{L}Z] = \frac{1}{N-1}\sum_{[\alpha,\beta]\in\mathcal{E}_N} \mathrm{Tr}\,\sigma_{\mathbf{k}}[|L_{\alpha,\beta}Z|^2] \ ,$$

and where

$$C_{\mathbf{k}} = \max\{\, T_{\mathbf{k}} \ , \ (\log d/4 + (\log 3/2)/r^2 \ \} + (\log 2)/2 \ .$$

Note that the constant $C_{\mathbf{k}}$ depends on $\mathbf{k}$ only through $r(\mathbf{k})$ and not on N.

Thank you for your Interest!