The generalized Birman-Schwinger principle

Jussi Behrndt (TU Graz)

with A.F.M. ter Elst (Auckland, NZ) and F. Gesztesy (Waco, US)

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PART I

The classical Birman-Schwinger principle

Laplacians and Schrödinger operators

Consider Laplacian (unperturbed/free Schrödinger operator)

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- More assumptions on $V \Rightarrow$ more/better properties of $\sigma(H)$

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Konno Kuroda'66, Gestzesy Latushkin Mitrea Zinchenko'05

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$$|\lambda_0| \le \|V\|_{L^1}^2, \qquad \sqrt{2}|\lambda_0| \le \sqrt{|\lambda_0| + |\operatorname{Re}\lambda_0|}\,\|V\|_{L^1}$$

PART II

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(i) If $\lambda_0 \in \sigma_p(H)$ and f_0 eigenfunction

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An abstract Birman-Schwinger principle

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PROBLEM: BS-principle for generalized ev's in $\ker(H - \lambda_0)^k$

PART III

The Birman-Schwinger principle for generalized eigenvectors (Jordan chains)

Jordan chains of functions and operators

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Main theorem

Let H_0 and $H = H_0 + V_2^* V_1$ as above, and $\lambda_0 \in \rho(H_0)$. Then

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and for $m = 1, \dots, k$
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...this is the classical theorem by Konno and Kuroda

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- Abstract version for operators in Krein spaces by Derkach

PART IV

Robin realisations of elliptic PDO's

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For $B : H^{1/2}(\mathcal{C}) \to H^{-1/2}(\mathcal{C})$ define Robin realisation in $L^2(\Omega)$

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More precisely, A_B is the *m*-sectorial operator associated to

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which is closed and sectorial with $\operatorname{dom} \mathfrak{a} = H^1(\Omega)$.

Dirichlet-to-Neumann map

Consider Dirichlet realization

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For $\varphi \in H^{1/2}(\mathcal{C})$ and $\lambda \in \rho(A_D)$ there exists unique $f_\lambda \in H^1(\Omega)$ such that

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Recall $\lambda \mapsto D(\lambda)$ holomorphic operator function on $\rho(A_D)$.

Jordan chains and the Birman-Schwinger principle

Theorem

Let A_B Robin realisation of $\mathcal{A}$ in $L^2(\Omega)$ and $\lambda_0 \in \rho(A_D)$.

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and $f_{-1} = 0$.

The special case $k = 0$

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Corollary

Let A_B Robin realisation of $\mathcal{A}$ in $L_2(\Omega)$ and $\lambda_0 \in \rho(A_D)$.

- If f_0 eigenvector of A_B at λ_0 then

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- If $D(\lambda_0)\varphi_0 = B\varphi_0$ and $\varphi_0 \neq 0$ then unique solution $f_0 \in H^1(\Omega)$ of

$$(\mathcal{A} - \lambda_0) f_0 = 0, \qquad f_0|_{\mathcal{C}} = \varphi_0,$$

  is an eigenvector of A_B at λ_0.

Thank you for your attention